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# Wilson Lines and Webs in Higher-Order QCD

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**Abstract** Wilson lines have a number of uses in non-abelian gauge theories. A topical example in QCD is the description of radiation in the soft or collinear limit, which must often be resummed to all orders in perturbation theory. Correlators involving a pair of Wilson lines are known to exponentiate in terms of special Feynman diagrams called “webs”. I will show how this language can be extended to an arbitrary number of Wilson lines, which introduces novel new combinatoric structures (web mixing matrices) of interest in their own right. I will also summarise recent results obtained from applying this formalism at three-loop order, before concluding with a list of open problems.

## 1 Introduction

It is well-known that scattering amplitudes in perturbative quantum field theory are beset by infrared (IR) divergences. Feynman rules tell us that we must integrate over all positions of radiated fields, and the integrand is such that a divergence occurs (in four spacetime dimensions) as we integrate out to infinity. In momentum space, this corresponds to a low energy, or *soft* gluon, as distinguished from the *hard* particles that emit the radiation. This hand-wavy explanation shows us that infrared divergences are not unique to QCD, but occur in many different (non)-abelian gauge theories, including gravity.

Infrared singularities have been studied for many decades, and it is known that they formally cancel when real and virtual diagrams are combined in suitably inclusive observables. However, large kinematic contributions remain after this cancellation, that frequently need to be summed up to all orders in perturbation theory in order to achieve meaningful comparison of theory to data. This is called *resummation*, and we are always trying to make this more precise. There are also more formal applications of IR singularities: they give us information about the all-order structure of perturbative quantum field theory, and are thus useful in understanding the underlying structure of both Yang–Mills theory, and its supersymmetric extensions. Furthermore, IR singularities are related to Wilson lines, which occur in many contexts, including the transverse-momentum dependent parton distributions (TMDs) discussed elsewhere at LightCone 2017. Thus, the study of IR divergences remains a highly active subject involving various branches of high energy physics.

In order to classify the structure of IR divergences further, let us first note that the structure of an arbitrary amplitude  $\mathcal{A}$  for the production of  $n$  partons has the schematic form (see e.g. ref. [1])

$$\mathcal{A} = \mathcal{H} \cdot \mathcal{S} \cdot \prod_{i=1}^n \frac{J_i}{\mathcal{J}_i}. \quad (1)$$

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Here  $\mathcal{H}$  is a process-dependent *hard function* that is infrared finite, and  $S$  the so-called *soft function* collecting all soft singularities. Associated with each external parton leg  $i$  is a *jet function* collecting collinear singularities. Finally, the *eikonal jet functions*  $\mathcal{J}_i$  correct for the double counting of radiation which is both soft and collinear. The soft function has an operator definition as a vacuum expectation value

$$S = \langle 0 | \Phi_1 \dots \Phi_n | 0 \rangle, \quad (2)$$

involving gauge-covariant *Wilson line operators*

$$\Phi_i = \mathcal{P} \exp \left[ i \mathbf{T}^a \int_{\mathcal{C}} dx^\mu A_\mu \right], \quad (3)$$

where  $\mathcal{P}$  denotes path ordering of the colour generators  $\mathbf{T}^a$  along the Wilson line contour  $\mathcal{C}$ , coinciding with the classical trajectory of each hard parton. Equation (2) can be intuitively understood as follows. Soft singularities arise from each hard parton emitting radiation with zero 4-momentum, such that it does not recoil, but follows its classical trajectory. The hard particles can still change by a phase, and for this phase to have the right gauge transformation properties to be part of an amplitude, it must be described by a Wilson line.

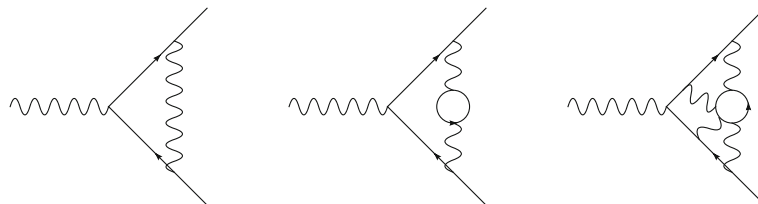
We thus see that classifying IR divergences amounts to calculating Feynman diagrams for multiple Wilson lines meeting at a point (where the latter corresponds to the hard interaction that produces the external partons). One may go further than this in showing that the soft function has an exponential form. That is, one may write

$$S \sim \exp \left[ \sum_W W \right], \quad (4)$$

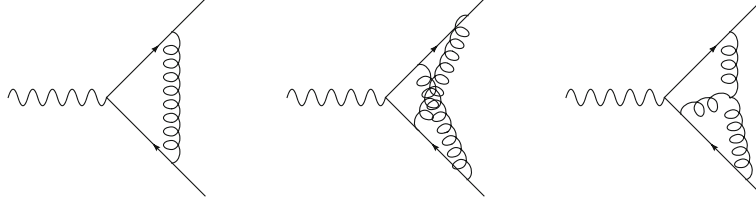
where the exponent (i.e. the logarithm of the soft function) contains a sum over a set of special Feynman diagrams called *webs*. This by itself looks like an empty statement—any function can be written as the exponential of its own logarithm. What makes this useful, however, is that we are able to develop methods for computing the logarithm of  $S$  *directly*, greatly simplifying the classification of IR singularities at higher orders.

The nature of webs depends upon which theory we are talking about. Let us first consider the relatively simple case of QED. As follows from the pioneering study of ref. [2], in this case the only diagrams entering the logarithm of the soft function have *connected subdiagrams*. That is, the part of the diagram representing the soft part of the amplitude (that remains upon removing the hard external lines) is connected. The first few “QED webs” are shown in Fig. 1, for the case of two Wilson lines (corresponding to the hard fermion-antifermion pair). However, the connected subdiagram property generalises straightforwardly to any number of Wilson lines. Note that, in the absence of propagating fermions, the logarithm of the soft function terminates at one-loop order, a property shared by other theories including quantum gravity [3–6].

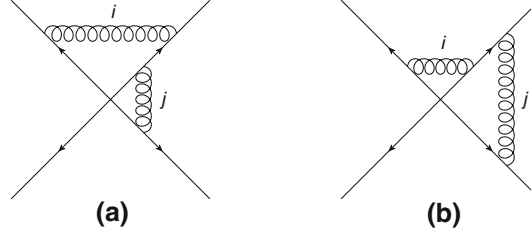
We have seen that webs in QED are relatively simple. Things are more complicated in QCD, due to the fact that emission vertices for soft gluons carry non-commuting colour matrices. However, one may still show that the soft function exponentiates in terms of webs. In order to describe the latter, it is useful to draw a distinction between the case of two Wilson lines meeting at a point, and more than two. For two lines, webs can be classified as the set of *irreducible* subdiagrams [7–9], where examples are shown in Fig. 2. In all such diagrams, shrinking one gluon to the origin automatically carries all other gluons in with it. Note that this can include non-connected subdiagrams, in contrast to the QED case. Furthermore, each web  $W$  turns out to have a modified colour factor  $\tilde{C}(W)$ , such that the soft function can be written



**Fig. 1** QED webs consist of connected subdiagrams



**Fig. 2** Irreducible QCD webs for two Wilson lines meeting at a point



**Fig. 3** A single web occurring at two-loop order

$$\mathcal{S} \sim \exp \left[ \sum_W \tilde{C}(W) \mathcal{F}(W) \right]. \quad (5)$$

Indeed, we can think of these so-called *exponentiated colour factors* as picking out which diagrams contribute to the exponent (i.e.  $\tilde{C}(W) = 0$  for a non-web).

Webs for multiple Wilson lines have only been studied relatively recently [10–12], and exhibit extra complication due to the fact that the soft function becomes matrix-valued in the space of possible colour flows. Indeed, the results indicate that webs are *closed sets* of diagrams rather than single ones, where the members of each set can be reducible. As an example, consider the two diagrams of Fig. 3 that occur at two-loop order, and which form a closed set under permutations of the gluon attachments on the Wilson lines. Each diagram  $D$  has a kinematic factor  $\mathcal{F}(D)$  and exponentiated colour factor  $\tilde{C}(D)$ , such that the contribution to the logarithm of the soft function is

$$\begin{pmatrix} \mathcal{F}(a) \\ \mathcal{F}(b) \end{pmatrix}^T \begin{pmatrix} \tilde{C}(a) \\ \tilde{C}(b) \end{pmatrix} = \begin{pmatrix} \mathcal{F}(a) \\ \mathcal{F}(b) \end{pmatrix}^T \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} C(a) \\ C(b) \end{pmatrix}, \quad (6)$$

where on the right-hand side we have used the fact that the exponentiated colour factors turn out to be expressible as a superposition of the normal colour factors  $C(D)$  of both diagrams. We see that the diagrams mix, such that colour and kinematic information is entangled in a non-trivial way. It thus makes sense to consider the set of two diagrams as single web, an interpretation that is further backed up by studying the renormalisation properties of multiparton webs [13]. Associated with the web is a *web mixing matrix*, that describes the entanglement of colour/kinematics.

This structure generalises: at arbitrary loop number, and for any number of Wilson lines, one may always arrange diagrams into sets which are closed under permutations of gluon attachments to the Wilson lines. The contribution of each set to the exponent is

$$\sum_{D, D'} \mathcal{F}_D R_{DD'} C_{D'},$$

where  $R_{DD'}$  is the mixing matrix of each web, comprised of numerical constants. The study of webs (and thus IR singularities) is then equivalent to the study of these matrices. They encode a huge amount of physics: namely, how colour and kinematic information is entangled, at all orders in perturbation theory! What we would ideally like to do is fully classify the structure of web mixing matrices, based on the combinatorics of web diagrams alone. A number of interesting properties have already been established. Firstly, any row of any web mixing matrix sums to zero, a property related to consistency with the abelian theory. A second property is *idempotence*:  $R^2 = R$ . Thus, web mixing matrices act as projection operators, allowing only certain

combinations of diagrams to contribute to the logarithm of the soft function. It is now known that all such combinations have *connected colour factors* [14] (i.e. superpositions of colour factors  $C(D)$  that resemble the colour factors of fully connected webs). This generalises a property noted already in the two-line case [7–9], and is known as *non-Abelian exponentiation*.

The proof of these properties rely on statistical physics methods (the *replica trick*), as well as known results from enumerative combinatorics [15]. More recently, it has been shown that the combinatorics of webs is related to that of partially ordered sets, or *posets* [16–18], which has generated new results in pure mathematics.

As well as classifying web mixing matrices, we also need to calculate the kinematic parts of web diagrams (see e.g. ref. [19] and references therein). The key quantity we wish to calculate is called the *soft anomalous dimension*  $\Gamma_S$ , which controls the ultraviolet singularities of Wilson lines. Recently, this quantity was calculated in QCD at three-loop order for the special case of massless particles [20]. The most complicated diagram is the fully connected gluon web, which took a few years to calculate, involving highly complex results at intermediate stages. However, the final result has a remarkably simple form if expressed in the right way. First, we note that the  $m$ -loop soft anomalous dimension for  $n$  particles can be written [21, 22]

$$\Gamma_S = \Gamma_S^{\text{dip.}} + \Delta_n^{(m)}, \quad (7)$$

where the first term depends only on pairs of particles, and the correction term starts at three-loop order. If  $\beta_i$  is the 4-velocity of the  $i^{\text{th}}$  Wilson line and  $\mathbf{T}_i$  a colour generator on line  $i$ , then the three-loop result is

$$\begin{aligned} \Delta_n^{(3)}(\{\rho_{ijkl}\}, \{\mathbf{T}_i\}) = 16 f_{abe} f_{cde} \left\{ -C \sum_{i=1}^n \sum_{\substack{1 \leq j < k \leq n \\ j, k \neq i}} \{\mathbf{T}_i^a, \mathbf{T}_i^d\} \mathbf{T}_j^b \mathbf{T}_k^c \right. \\ + \sum_{1 \leq i < j < k < l \leq n} \left[ \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d \mathcal{F}(\rho_{ikjl}, \rho_{iljk}) + \mathbf{T}_i^a \mathbf{T}_k^b \mathbf{T}_j^c \mathbf{T}_l^d \mathcal{F}(\rho_{ijkl}, \rho_{ilkj}) \right. \\ \left. \left. + \mathbf{T}_i^a \mathbf{T}_l^b \mathbf{T}_j^c \mathbf{T}_k^d \mathcal{F}(\rho_{ijlk}, \rho_{iklj}) \right] \right\}, \quad (8) \end{aligned}$$

where  $C = \zeta_5 + 2\zeta_2\zeta_3$ , and  $\mathcal{F}$  is a function of *conformally invariant cross-ratios*  $\rho_{ijkl} = \frac{(\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)}$ . The non-trivial kinematic dependence is simplified by introducing the (in general complex) variables

$$z_{ijkl} \bar{z}_{ijkl} = \rho_{ijkl}, \quad (1 - z_{ijkl})(1 - \bar{z}_{ijkl}) = \rho_{ilkj}. \quad (9)$$

Then one has

$$\mathcal{F}(\rho_{ijkl}, \rho_{ilkj}) = F(1 - z_{ijkl}) - F(z_{ijkl}), \quad (10)$$

where

$$F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2[\mathcal{L}_{001}(z) + \mathcal{L}_{100}(z)], \quad (11)$$

and  $\mathcal{L}_w(z)$  is a single-valued harmonic polylogarithm [23] (SVHPL). The simplicity of this result suggests an alternative way to calculate it, namely to use a *bootstrap* procedure. By mapping Wilson lines to the Riemann sphere, one can show that the three-loop correction function  $\Delta_n^{(3)}$  can only depend on SVHPLs. One can then write a general ansatz for  $C$  and  $F(z)$ , and constrain the coefficients using (1) Bose symmetry; (2) colour conservation; (3) uniform transcendental weight (motivated by a conjecture for the closely related result in  $\mathcal{N} = 4$  Super-Yang–Mills theory); (4) collinear limits; (5) Regge limits (using information from ref. [24]). As shown in ref. [25], the form of the correction function is then completely fixed up to an overall constant. Furthermore, this is the first time such a procedure has been used in a fully non-planar QCD application!

To summarise, IR singularities are important for both *hep-ph* and *hep-th* reasons. There has been much recent progress in calculating the soft anomalous dimension that controls them. Webs provide an efficient language for higher order calculations in QCD and related theories. Finally, new powerful techniques (e.g. the bootstrap approach) greatly simplify calculations at three loops, and potentially beyond. A number of open questions remain:

- Can we classify the general structure of web mixing matrices?
- Are their combinatoric properties useful for something else (e.g. ref. [16] hints at intriguing applications in computer science)?
- Can we calculate all 3-loop Wilson line diagrams for massive particles?
- Can we use the bootstrap at four loops and beyond?
- Can we use these results in phenomenological applications (e.g. resummation)?
- Can webs be extended to beyond the soft approximation (refs. [26,27] suggest this is the case)?
- Are webs useful for TMDs, GPDs or other subjects discussed at LightCone 2017?

The answer to many of these questions is very likely to be yes!

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